ON SOME CLASS OF MULTIDIMENSIONAL NONLINEAR INTEGRABLE SYSTEMS¹

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On the base of Lie algebraic and differential geometry methods, a wide class of multidimensional nonlinear integrable systems is obtained, and the integration scheme for such equations is proposed.

1. In the report we give a Lie algebraic and differential geometry derivation of a wide class of nonlinear integrable systems of partial differential equations for the functions depending on an arbitrary number of variables, and construct, following the lines of Refs. 1–3, their general solutions in a 'holomorphically factorisable' form. The systems are generated by flat connections, constrained by the relevant grading condition, with values in an arbitrary reductive complex Lie algebra $\mathcal G$ endowed with a $\mathbf Z$ -gradation. They describe a multidimensional version of Toda type fields coupled to matter fields, and, analogously to the two dimensional situation, with an appropriate Inönü–Wigner contraction procedure, for our systems one can exclude back reaction of the matter fields on the Toda fields.

For two dimensional case and the connection taking values in the local part of a finite dimensional algebra \mathcal{G} , our equations describe an (abelian and nonabelian) conformal Toda system and its affine deformations for an affine \mathcal{G} , see Ref. 1 and references therein, and also Ref. 2 for differential and algebraic geometry background of such systems. For the connection with values in higher grading subspaces of \mathcal{G} one deals with systems discussed in Refs. 3, 4. In higher dimensions our systems, under some additional specialisations, contain as particular cases the Cecotti–Vafa type equations⁵ written there for a case of the complexified orthogonal algebra, see also Ref. 6; and those of Gervais–Matsuo⁷ which represent some reduction of a generalised WZNW model. Due to the lack of space we present here only an announcement of the results which will be described in detail, together with some remarkable examples elsewhere.

2. Let M be the manifold \mathbf{R}^{2A} with the standard coordinates $z^{\pm i}, 1 \leq i \leq A$, or \mathbf{C}^A with $z^{+i} = \overline{z^{-i}}$; let G be a reductive complex Lie group with the Lie algebra \mathcal{G} endowed with a \mathbf{Z} -gradation, $\mathcal{G} = \bigoplus_{m \in \mathbf{Z}} \mathcal{G}_m$. Consider a flat connection $\omega = \sum_{i=1}^{A} (\omega_{-i} dz^{-i} + \omega_{+i} dz^{+i})$ on the trivial principal fiber bundle

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 $M \times G \to M$, and impose on it the grading condition that the components $\omega_{\pm i}$ take values in $\mathcal{G}_0 \oplus \mathcal{N}_{\pm i}$, where $\mathcal{N}_{\pm i} = \bigoplus_{1 \leq m \leq l_{\pm i}} \mathcal{G}_{\pm m}$ with some positive integers $l_{\pm i}$, such that the subspaces $\mathcal{G}_{\pm l_{\pm i}}$ are nontrivial. Restrict also the connection components $\omega_{\pm i} = \sum_{m=0}^{\pm l_{\pm i}} \omega_{\pm i,m}$ by the condition $\omega_{\pm i,\pm l_{\pm i}} = \zeta_{\pm} c_{\pm i} \zeta_{\pm}^{-1}$, where ζ_{\pm} are some mappings $M \to G_0$ with G_0 being the Lie group corresponding to \mathcal{G}_0 , and $c_{\pm i}$ are some fixed elements of the subspaces $\mathcal{G}_{\pm l_{\pm i}}$ satisfying the relations $[c_{\pm i}, c_{\pm j}] = 0$. Then one can prove the following statement.

There exists a local G_0 -gauge transformation that brings a connection satisfying the above given conditions to the connection ω with the components

$$\omega_{+i} = \gamma^{-1} \left(\sum_{m=1}^{l_{+i}-1} v_{+i,m} + c_{+i} \right) \gamma,$$

$$\omega_{-i} = \gamma^{-1} \partial_{-i} \gamma + \sum_{m=-1}^{l_{-i}+1} v_{-i,m} + c_{-i},$$

where γ is some mapping from M to G_0 , and $v_{\pm i,m}$ are mappings taking values in $\mathcal{G}_{\pm m}$.

The equations for the mappings γ and $v_{\pm i,m}$ which follow from the flatness condition we call multidimensional Toda type systems, and the corresponding functions parametrising the mappings γ and $v_{\pm i,m}$ — Toda and matter type fields, respectively. In the proof we use the so called modified Gauss decompositions² which allow to overcome the main disadvantage of any standard Gauss decomposition that not any element of G possesses this decomposition. Namely, if an element $a \in G$ does not admit the Gauss decomposition of some form, then subjecting a to a left shift in G we can get an element having this decomposition.

Let us give examples of multidimensional Toda type equations, namely those corresponding to the cases $l_- = l_+ = l = 1, 2$. For l = 1 one has

$$[c_{\pm i}, \gamma^{\pm 1} \partial_{\pm j} \gamma^{\mp 1}] - [c_{\pm j}, \gamma^{\pm 1} \partial_{\pm i} \gamma^{\mp 1}] = 0,$$

$$\partial_{\pm i} (\gamma^{-1} \partial_{-i} \gamma) = [c_{-i}, \gamma^{-1} c_{\pm i} \gamma].$$
(2)

For l=2 with the renotation $v_{\pm i,\pm 1} \equiv v_{\pm i}$ one has

$$[c_{\pm i}, v_{\pm j}] = [c_{\pm j}, v_{\pm i}],$$

$$\partial_{\pm i}v_{\pm j} \pm [\gamma^{\pm 1}\partial_{\pm i}\gamma^{\mp 1}, v_{\pm j}] = \partial_{\pm j}v_{\pm i} \pm [\gamma^{\pm 1}\partial_{\pm j}\gamma^{\mp 1}, v_{\pm i}],$$

$$[c_{\pm i}, \gamma^{\pm 1}\partial_{\pm j}\gamma^{\mp 1}] - [c_{\pm j}, \gamma^{\pm 1}\partial_{\pm i}\gamma^{\mp 1}] + [v_{\pm i}, v_{\pm j}] = 0;$$

$$\partial_{\pm i}v_{\mp j} = [c_{\mp j}, \gamma^{\mp 1}v_{\pm i}\gamma^{\pm 1}],$$

$$\partial_{+j}(\gamma^{-1}\partial_{-i}\gamma) = [v_{-i}, \gamma^{-1}v_{+j}\gamma] + [c_{-i}, \gamma^{-1}c_{+j}\gamma].$$